Computing multiple solutions of PDEs

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September 30, 2020

Section 1

Introduction

Can you conduct an experiment twice . . .

and get two different answers?

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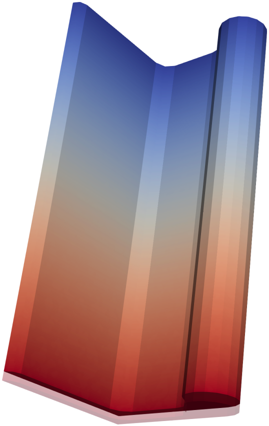
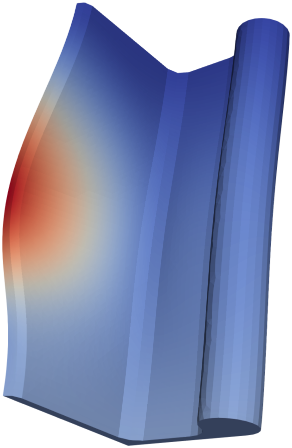


Axial displacement test of an Embraer aircraft stiffener.

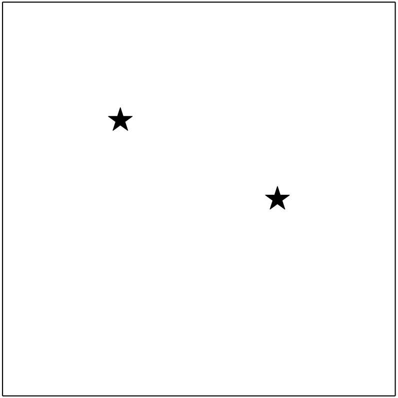
Can you conduct an experiment twice . . .

and get two different answers?



Two different, stable configurations.

Why worry?



A PDE with two unknown solutions

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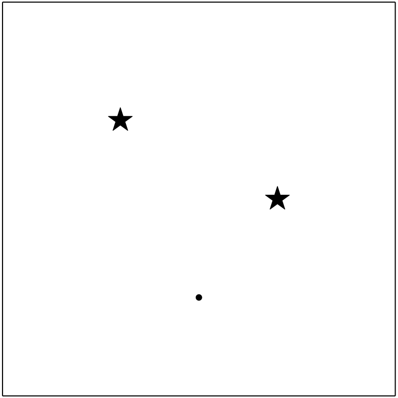

Start from some initial guess

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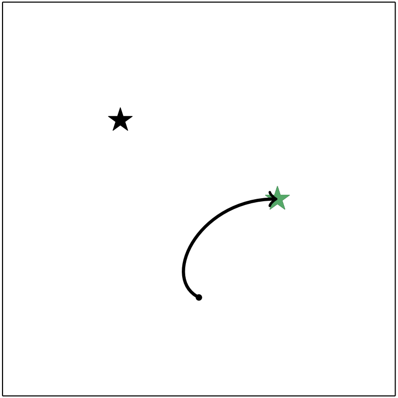

We converge to one solution, our prediction

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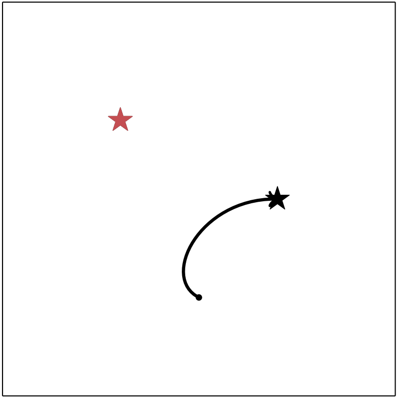

But nature has chosen another (unknown) solution!

Mathematical formulation

Compute the multiple *solutions* u of an equation

$$f(u, \lambda) = 0$$
$$f : V \times \mathbb{R} \to V^*$$

as a function of a parameter λ.

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What is f?

f could represent stationary states, periodic orbits, bifurcation points in another parameter, . . .

Applications

Navier–Stokes, hyperelasticity, topology optimisation, liquid crystals, Bose–Einstein condensates, . . .

Section 2

Deflation

The core idea

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Fix parameter λ. Given

- a Fréchet differentiable residual $\mathcal{F} : V \to V^*$
- a solution $r \in V$, $\mathcal{F}(r) = 0$, $\mathcal{F}'(r)$ nonsingular

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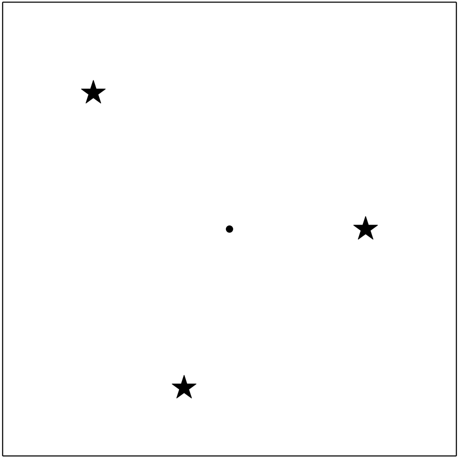
Find more solutions, starting from the same initial guess.

Finding many solutions from the same guess



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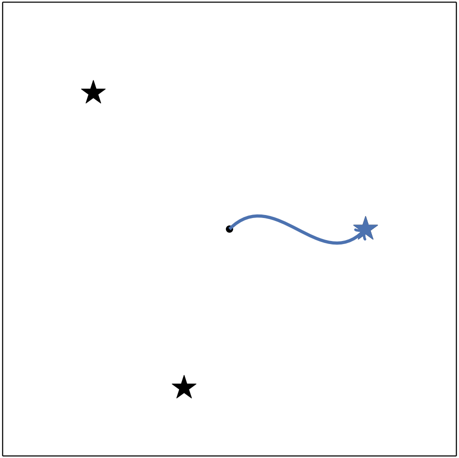
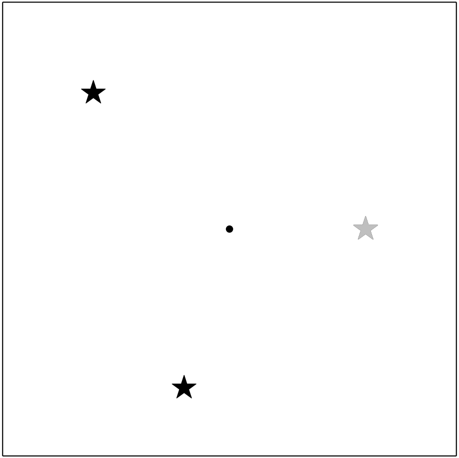


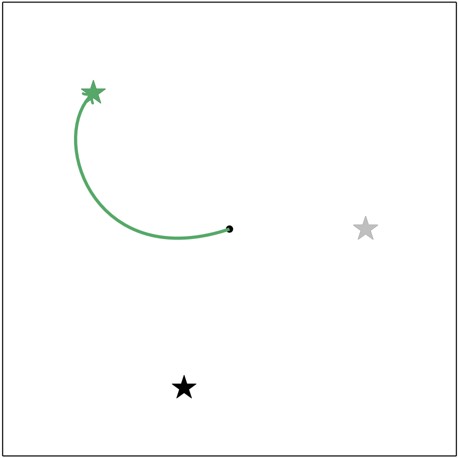
Step I: Newton from initial guess

Finding many solutions from the same guess



Step II: deflate solution found

Finding many solutions from the same guess



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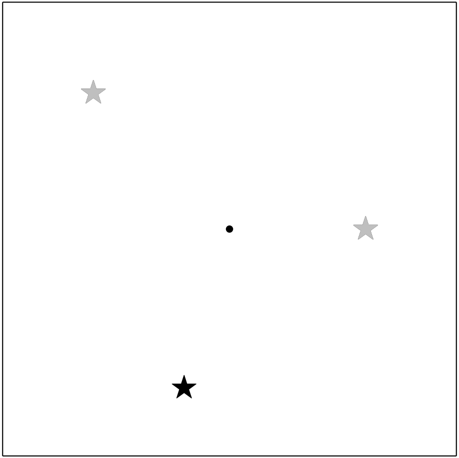

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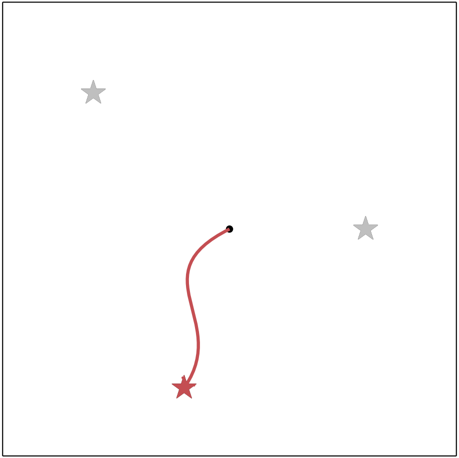

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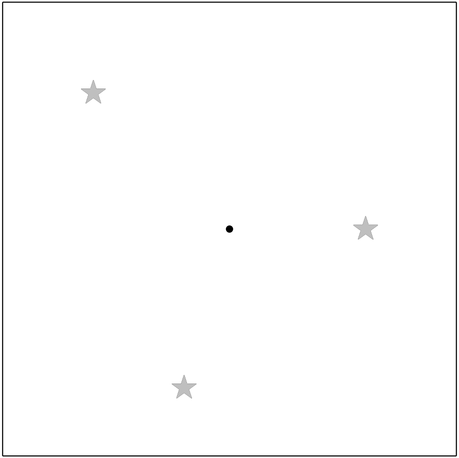

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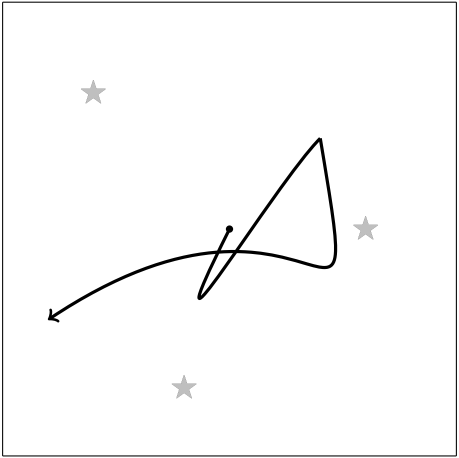

Step III: termination on nonconvergence

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Construction of deflated problems

A nonlinear transformation

$$\mathcal{G}(u) = \mathcal{M}(u; r)\mathcal{F}(u)$$

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A deflation operator

We say $\mathcal{M}(u; r)$ is a deflation operator if for any sequence $u \to r$

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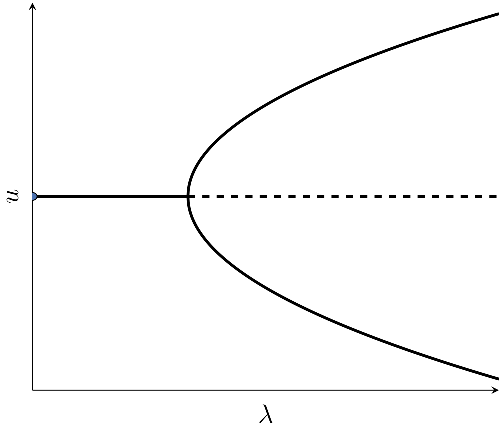
Theorem (F., Birkisson, Funke, 2014)

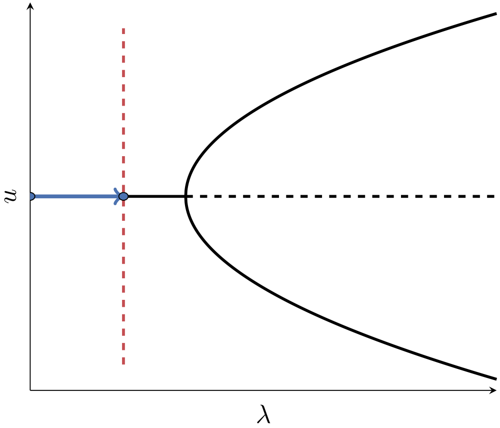
This is a deflation operator for $p \geq 1$:

$$\mathcal{M}(u; r) = \left(\frac{1}{\|u - r\|^p} + 1 \right)$$

Deflated continuation

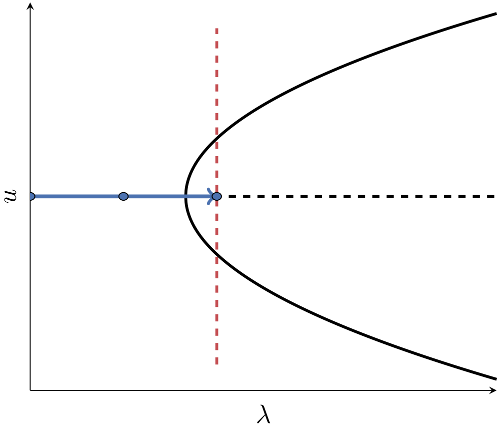


Deflated continuation



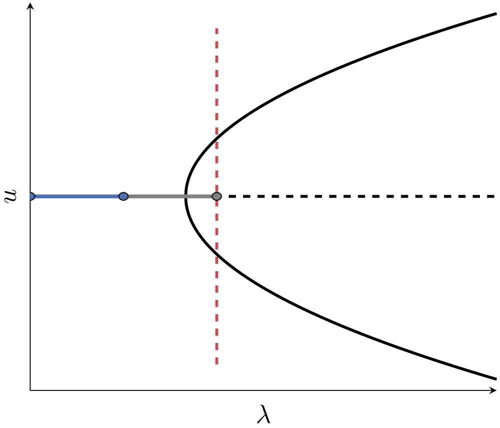
Step I: continuation

Deflated continuation



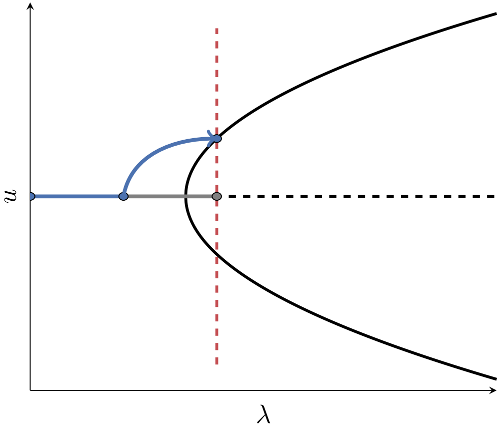
Step II: continuation

Deflated continuation



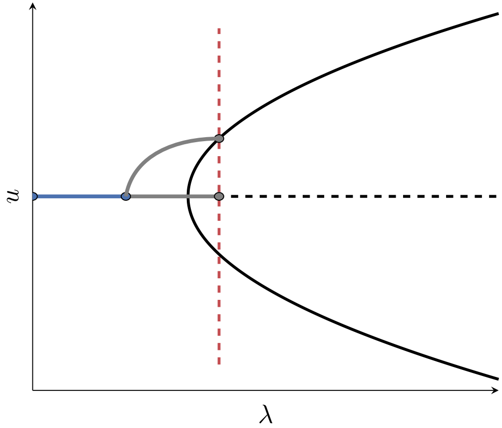
Step III: deflate

Deflated continuation



Step III+: solve deflated problem

Deflated continuation



Step III: deflate

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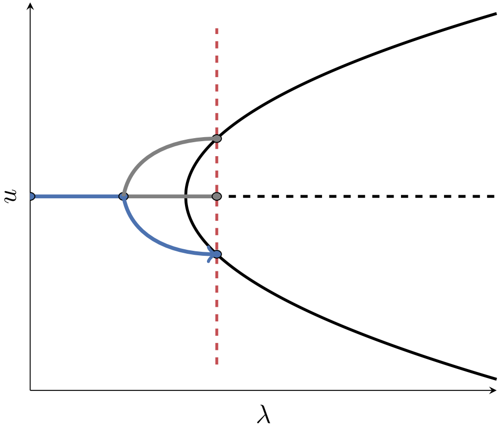

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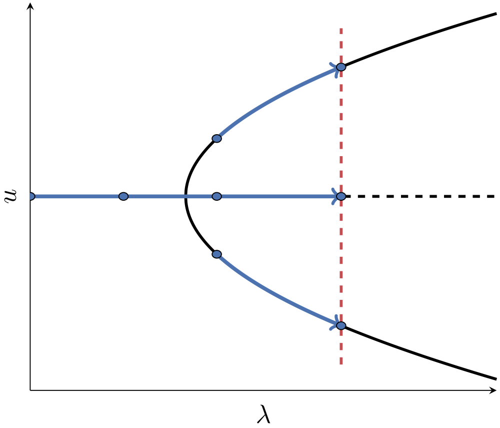
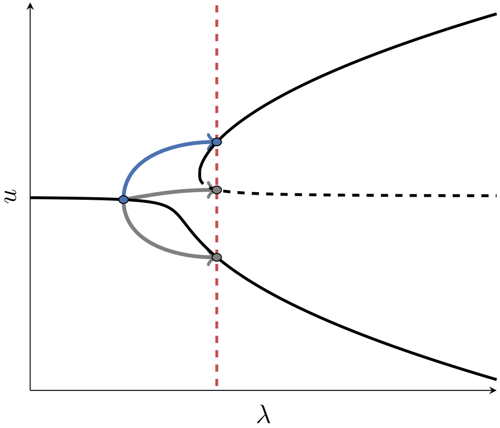


Step IV: continuation on branches

Deflated continuation



A disconnected diagram.

An example: the winged cusp

The winged cusp for $x \in \mathbb{R}$

$$f(x, \lambda) = x^3 - 2\lambda x + \lambda^2 - 2\lambda + 1 = 0$$

Section 3

Applications

Application: Bose–Einstein condensates

Stationary Gross–Pitaevskii equation

$$-\frac{1}{2}\Delta\phi + \frac{x^2+y^2+z^2}{2}\phi - \mu\phi + |\phi|^2\phi = 0, \qquad \phi|_{\partial\Omega} = 0.$$

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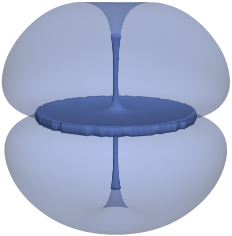

Solutions for $\mu = 6$. A vortex line and a planar dark soliton.

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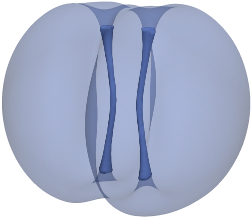

Solutions for $\mu = 6$. A pair of vortex lines.

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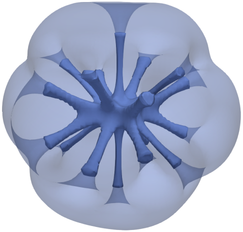

Solutions for $\mu = 6$. A vortex star.

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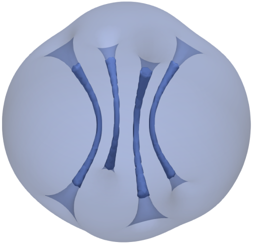

Solutions for $\mu = 6$. Four vortex lines of alternating charge.

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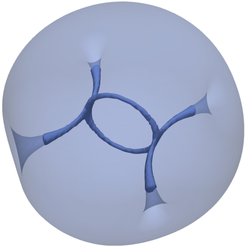

Solutions for $\mu = 6$. A vortex ring with two "handles".

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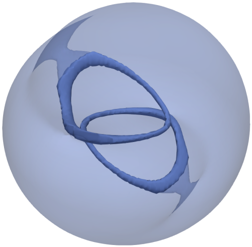

Solutions for $\mu = 6$. Two bent vortex rings?

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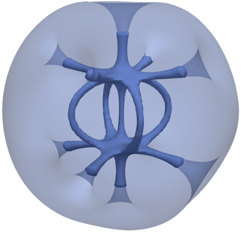

Solutions for $\mu = 6$. Two vortex rings and five lines?

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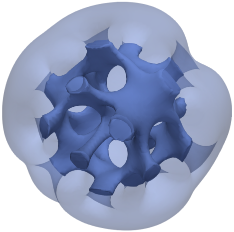

Solutions for $\mu = 6$. A vortex ring cage?

Conclusions

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- Deflation is a useful technique for finding them.
- Deflated problems can be **solved efficiently**.